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Optimal value at risk hedge using simulation methods

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Abstract (Document Summary)

This paper proposes simulation methods to **calculate value** at **risk** measures for a trading portfolio. This approach allows for an optimization of dynamic hedges by attempting to minimize the **risk** of the hedge through time. Its applicability is shown with examples from Latin American emerging stock and **derivatives** markets, which are some of the most challenging in the world in terms of volatility and liquidity.

Full Text (3193 words)

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A financial institution acting as a market maker, originator, or trader must be careful to prevent losses in its derivatives books. For instance, the hedge of options portfolios may not be a simple task (Boyle and Emanuel [1980], Galai [1983], and Hull and White [1987]). Consider a financial institution that sold exotic options on a stock index. In this case, the option's sensitivity to the underlying (stock index) changes continuously as time passes. Also, the price of the exotic option is sensitive to other factors such as interest rates and the index's volatility. Since the exposures to risk factors such as the underlying stock index, interest rates, and the index's volatility change continuously, it is necessary to rebalance the hedge frequently.

Liquidity and the availability of hedging instruments are two other important problems that need to be considered. Take the example of exotics sold on the Mexican stock index, Indice de Precios y Cotizaciones (IPC), and assume

the writer (of the option) wants to hedge his exposure using only instruments traded in American exchanges to minimize currency, operational, and sovereign risks. ADRs on Mexican stocks (available at the New York Stock Exchange), futures on the IPC (available at the Chicago Mercantile Exchange), and options on ADRs on Telefonos de Mexico (available at the Chicago Board of Options Exchange) can be used to control the delta, gamma, rho, and kappa risks of a position. However, few among these instruments are liquid and can be used in a dynamic hedging strategy: some ADRs and one (or at the most two) options on ADRs on Telefonos de Mexico.

There are several approaches to derivatives hedging. Three of these approaches are covered positions, stop-loss strategies, and dynamic strategies. Dynamic hedging strategies play an important role when managing the risks of options portfolios. In this article, we concentrate on dynamic hedging strategies.

Value at risk (VaR) has gained acceptance in world financial markets as the most appropriate risk measure. It measures the worst-case expected loss of a portfolio over a given holding period (say, one day) at a specified confidence level (say, 99%). For instance, a portfolio whose VaR is $10 million over a one-day holding period, with a 99% confidence level, would have only a 1% probability of suffering an overnight loss greater than $10 million. Among the methodologies used to estimate the VaR of a portfolio, simulation methods (historical and Monte Carlo) are the best choices, especially when portfolios with options or instruments with embedded options are analyzed (Jorion [1997]).

Optimization is an important mathematical technique when hedging dynamically options portfolios. However, no portfolio optimization methodology in the finance literature incorporates the VaR estimate using simulation methods as its risk measure. This article proposes an optimization methodology for the computation of hedges that minimize the VaR estimated using simulation methods. This methodology is an essential part of the proposed dynamic hedging strategy, and its use is convenient because:

1. It weighs the relative importance of the Greek risks (such as delta, gamma, rho, and kappa) to achieve the optimal hedge. This allows the hedger to explore in an optimal way the correlation between different Greek risks when obtaining the optimal hedge. Also, since all Greek risks are considered together during the optimization phase, traditional dynamic hedging strategies (such as delta hedging and delta-gamma hedging) are only particular cases of our proposal. This is particularly important when there are not many liquid instruments available for hedging, as in emerging derivatives markets.

2. The Value at Risk of a portfolio is minimized using a scenario-based optimization methodology (Koskosidis and Duarte [1997]). There are several advantages apparent when this methodology is compared with an analytic-based optimization methodology, such as Markowtiz's Mean-Variance (MV) (Markowitz [1959]). One is that scenariobased methodologies obviate the use of singlepoint forecasts for covariance and expected returns, as in the MV methodology. Also, the minimum variance hedging methodology (Johnson [1960]), which is derived from the MV methodology, can be obtained as a very particular case of our proposal, for a very specific choice of parameters (Chamberlain [1993], Epstein [1985], Kallberg and Ziemba [1983]). Another advantage is that any asymmetry of a portfolio's distribution of returns can be handled easily during the optimization phase. A third advantage is that since scenario-based methodologies handle multiple forecasts separately, they allow the hedger to "stress test" a final hedged portfolio during the optimization phase.

3. The Value at Risk of a portfolio, as measured by the risk management group, is minimized when obtaining the optimal hedge. The adoption of a hedging methodology that is consistent with the risk measurement methodology used by the risk management group is important because it reduces the chances of violating previously established Value at Risk limits. Also, our proposal helps the integration of risk management and trading systems by establishing a common risk measurement methodology.

4. The hedge of the gamma, kappa, and rho risks is very important in illiquid and very volatile markets, such as emerging markets. This is illustrated in this article by two examples that use data from Latin American emerging stock and derivative markets.

SIMULATION METHODS

There are basically two overall methodologies used to estimate the VaR of a portfolio: analytic and simulation Jorion [1997]). Among the analytic methods, the two most commonly used possibilities are the delta and the delta-gamma approximations. Among the simulation methods, the two possibilities are the historical and the Monte Carlo methods. For the same portfolio, the use of different methodologies can produce VaR estimates substantially

different.

Simulation methods combined with full valuation are the most appropriate for the risk analysis of portfolios presenting non-linearities (such as options and instruments with embedded options). A simple illustration is given in Exhibit 1 where the VaRs (for one week, at a 99% confidence level) are reported for a one-year atthe-money American put option, on one futures contract of the Sao Paulo Stock Exchange Index. (Futures contracts on the Sao Paulo Stock Exchange Index are negotiated in the Bolsa de Mercadorias e Futuros, Sao Paulo, Brazil.) Two positions are considered: long the put option and short the put option. Two methodologies for estimating the VaR are used: the delta equivalent analytic method, and the Monte Carlo simulation method with full valuation and ten thousand scenarios.

One observes that while the delta equivalent analytic approach provides equal VaR estimates for the long and short positions, the Monte Carlo simulation methodology estimates differ substantially. Exhibit 2 illustrates graphically why this is the case for the Monte Carlo simulation methodology: the smoothed estimate for the probability density of the option's expected return is asymmetric. This leads to a substantial difference when estimating the one percentile of the distributions in Exhibit 2 (in absolute value), which correspond to their VaR at a 99% confidence level. On the other hand, the delta equivalent analytic methodology assumes that the distribution of the put option's expected return presents the same shape as that of its underlying assets (the futures contract), which is symmetric. The consequence of this modeling hypothesis is illustrated in Exhibit 1: the VaR is the same for both positions (long and short). This is unacceptable from the modeling point of view.

EXHIBIT 1

Value at Risk for Two Positions Using Two Methodologies

Position	Methodology	Value at Risk (weekly, 99%)*
Long Put Option	Delta Equivalent Analytic	[illegible]
Short Put Option	Delta Equivalent Analytic	[illegible]
Long Put Option	Monte Carlo Simulation	[illegible]
Short Put Option	Monte Carlo Simulation	[illegible]

*[illegible]

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EXHIBIT

The use of portfolio optimization techniques (which include the optimal hedging problem as a particular case) that take into account asymmetries in the expected returns of derivatives portfolios is necessary. The dynamic hedging methodology described in the next section satisfies this requirement.

OPTIMAL VALUE AT RISK HEDGE

The optimal value at risk methodology is a dynamic hedging strategy that minimizes the VaR of a portfolio at each rebalancing period. That is, for a given initial portfolio and set of hedging instruments, the methodology can be used to rebalance the hedged portfolio on a continuous basis. It updates the optimal number of contracts to be bought/sold according to the latest price fluctuations in the market, minimizing the VaR of the hedged portfolio (initial portfolio plus hedging instruments).

The optimization model used in the dynamic hedging methodology is based on the following modeling and operational principles:

1. The use of the optimal VaR hedging methodology requires that scenarios be generated for each hedging instrument and for the portfolio. These scenarios should incorporate the latest price fluctuations in the market. They can be generated using standard simulation methods for VaR estimation (Monte Carlo or historical), or incorporate investors' opinions as described in Koskosidis and Duarte [19971.

2. Different hedging instruments (stocks, options, and futures) on different underlying assets can be used for hedging. The model allows the user to require that only round lots be bought/sold. Remember that since odd-lot trading is prohibitively expensive for use in dynamic hedging strategies, especially in emerging markets, the amount of each hedging instrument to be bought/sold should preferably avoid odd-lot trading. As an example we mention that while the estimated cost of a "round trip" (purchase and later sale) of a round lot of stocks by an American institutional investor is about 2.1% in Argentina and 1.7% in Brazil, these costs can easily double (even for liquid stocks) when odd-lot trading is used in these two countries.

3. Since the model is designed to be used continuously during trading hours, it is reliable, user-friendly,

computationally efficient, and easy to maintain, to avoid operational risk.

The rigorous mathematical formulation of the optimization problem is presented in the appendix. Also, implementation details important for those interested in reproducing our proposal in their daily routine are discussed in the appendix.

The next sections present two case studies that illustrate the use of the optimal VaR hedging methodology in practice. The two examples are taken from the Brazilian stock and derivatives markets.

Case Study: Hedging Put Options on Futures Contracts

Three exchanges offer options in Brazil: the Bolsa de Mercadorias and Futuros (stock indexes, foreign exchange, and short-term interest rates), the Sao Paulo Stock Exchange (stocks), and the Rio de Janeiro Stock Exchange (stocks).

Among the stock options available in the Sao Paulo Stock Exchange and the Rio de Janeiro Stock Exchange, only at-the-money and close-to-expiration options on Telebras PN are liquid and can be used in dynamic hedging strategies. Brazilian risk managers have few liquid options (usually two) available for use in dynamic hedging strategies. Under these circumstances, the hedger must weigh the relative importance of the Greek risks. The optimization model given in the appendix allows this weighing to be done in an optimal way.

Besides options, futures contracts on the Sio Paulo Stock Exchange Index are used as hedging instruments in the Brazilian stock market (Duarte and Mendes [1998]). These futures contracts are available for negotiation in the Bolsa de Mercadorias and Futuros. Although six series are always listed at the Bolsa de Mercadorias and Futuros, only those next-to-expire are liquid to be used in a dynamic hedging methodology. Futures contracts are also combined with indexed portfolios in hedging strategies in Brazil in order to exploit arbitrage opportunities between stock and derivatives markets (Duarte [1997]).

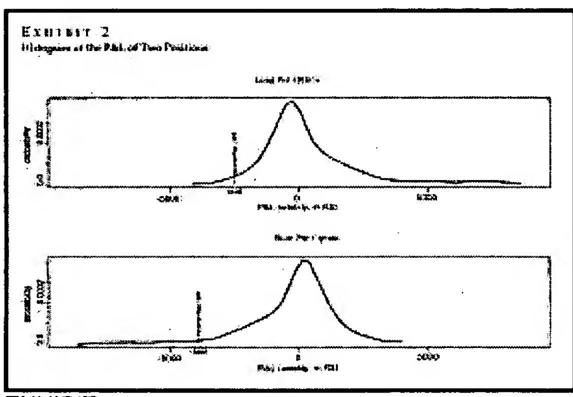

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EXHIBIT

In this (and the next) case study we shall use the only three liquid hedging instruments available in Brazilian derivatives markets:

1. next-to-expire futures contracts on the Sao Paulo Stock Exchange Index;

2. one at-the-money and close-to-expiration call

option on Telebras PN; and

3. one at-the-money and close-to-expiration put option on Telebras PN.

The following five characteristics of Brazilian financial markets illustrate why gamma, kappa, and rho risks should not be neglected:

1. The Brazilian stock market is much more volatile than more developed stock markets. For example, the GARCH volatility of the Sao Paulo Stock Exchange Index was consistently larger than the GARCH volatility of the New York Stock Composite Index during the years of 1995, 1996 and 1997; see Exhibit 3.

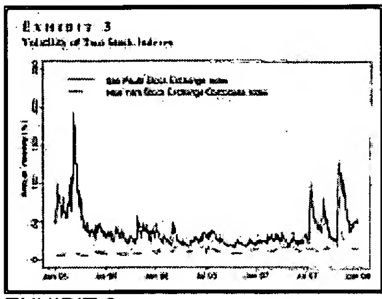

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EXHIBIT 3

2. The same is true when the Brazilian fixed-income market is compared to the American and European fixed-income markets. For example, the average GARCH yield volatility of the one-month rates of Brazilian CDs was approximately eight times larger than its corresponding value for the one-month LIBOR U.S. dollar during the years of 1995, 1996, and 1997. This illustrates the importance of the rho risk for options on local stocks and stock indexes.

3. The implied volatility by liquid Brazilian stock options varies much more than the implied volatility by liquid stock options negotiated at American and European exchanges. Remember that the variation of this implied volatility can be used to measure the kappa risk. The implied volatility of atthe-money liquid calls negotiated in the Sao Paulo Stock Exchange on Telebras PN are depicted in Exhibit 4. One observes that the implied volatility varies between 20% and 100% for the three years covered in Exhibit 4. This illustrates the importance of considering kappa risk for options on local stocks and stock indexes.

4. The gamma of plain vanilla calls/puts is larger for those options at-the-money and close-to-expiration. These are the only liquid options that can be used in dynamic hedging strategies in Brazilian derivatives markets. Ignoring the gamma risk of these options can lead to an underestimation of the market risk of a portfolio with options, as illustrated in Duarte and Maia [1997]. This shows the importance of gamma risk. Not considering the Greek risks (other than delta risk) when hedging in the Brazilian stock and derivatives markets results in model risk. We strongly recommend using more sophisticated dynamic hedging strategies, instead of simply delta hedging, when managing the market risk of derivatives portfolios in Brazil.

As a first illustration, let us assume that a financial institution sold one thousand one-year at-the-money American put options on futures contracts on the Sao Paulo Stock Exchange Index. This is the same option which the distribution of returns show in Exhibit 2. Ten thousand scenarios generated using Monte Carlo simulation with full valuation (as in Exhibit 1) are used. The VaR (weekly, 99%) for the unhedged portfolio with these one thousand options is R$3,892,360.21; see Exhibit 5.

The results of hedging methodologies using the three hedging instruments just mentioned, separately, are given in Exhibit 5. For example, the VaR of the hedged portfolio obtained using only futures contracts and the minimum variance hedging methodology Johnson [1960]) remains at R$1,609,079.27. Exhibit 5 shows that the optimal VaR hedging methodology provides better results when compared to the minimum variance. It also provides better results when compared to the results obtained using only one option with the delta hedge methodology.

Although the use of several hedging instruments provides better hedges, one must be careful to select which instruments should be combined. For instance, the combination of futures contracts and the put option on Telebras PN improves substantially the hedge, bringing the VaR of the portfolio to only R$1,053,836.42, almost one fourth of its initial exposure. However, the inclusion of the call option on Telebras PN presents only a minor effect on the hedge performance, bringing the VaR from R$1,053,836.42 to R$989,310.43.

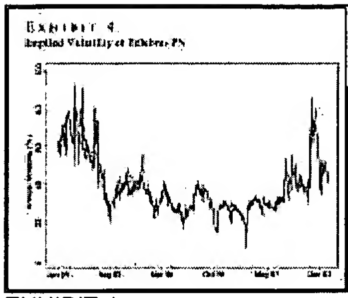

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EXHIBIT 4

Since the optimal VaR hedging methodology proposed admits a very reliable and efficient implementation from the computational point of view (see appendix), the hedger can easily experiment with several possibilities in a short time interval to obtain the most suitable combination of hedging instruments in his opinion.

The next example addresses the question of combining hedging instruments for the optimal VaR hedge of a portfolio with stocks.

Case Study: Hedging a Portfolio with Stocks

A R$60 million portfolio composed of six liquid Brazilian stocks is considered in this case study; see Exhibit 6. The same three hedging instruments considered in the previous case study are used. The Monte Carlo simulation methodology is used to generate ten thousand scenarios for this case study. A graphical representation of the portfolio's market exposure reduction, measured using the VaR (weekly, 95%), using different combinations of hedging instruments, is depicted in

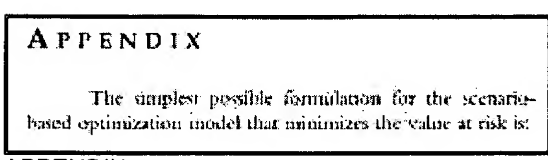
Exhibit 7. This case study illustrates how the hedging methodology proposed can be used to select the most satisfying combination of hedging instruments.

The unhedged portfolio (only stocks) presents a VaR of R$5,330,744.00. The optimal VaR methodology using futures contracts on the Sao Paulo Stock Exchange Index produces the most significant reduction in the portfolio's market exposure when using only one hedging instrument: a VaR of R$814,612.00. Combining hedging instruments pairwise produces another substantial reduction in the portfolio's market exposure: in the best case, the Value-at-Risk is brought down to R$527,767.00, when futures contracts are combined with at-themoney put options on Telebras PN. Finally, the use of all three hedging instruments together reduces the portfolio's VaR only marginally to R$479,810.00. Once more, including at-the-money call options on Telebras PN does not produce much better results than using only futures contracts and at-the-money put options on Telebras PN.

Choosing the most satisfying combination of hedging instruments can be very easily and efficiently done using the optimization model given in the appendix, as illustrated in Exhibit 7. For instance, it took less than two minutes to generate all the results in Exhibit 7, as explained in the appendix.

CONCLUSION

We present a dynamic hedging methodology that minimizes the VaR of portfolios using simulation techniques. This methodology is based on a reliable, computationally efficient, easy to implement and use optimization framework. Two examples from the Brazilian stock and derivatives markets are presented to illustrate its practical use. The methodology outlined is particularly useful when hedging portfolios in illiquid and very volatile markets, such as emerging derivatives markets.



APPENDIX

The simplest possible formulation for the scenario-based optimization model that minimizes the value at risk is:

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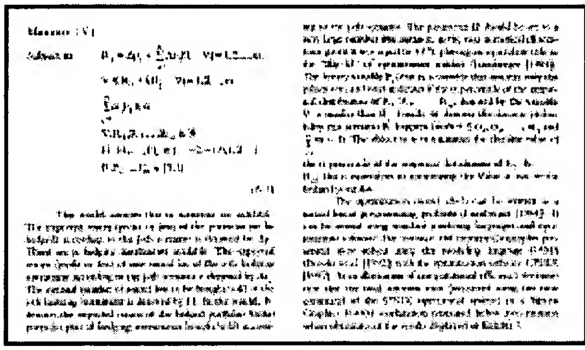
APPENDIX



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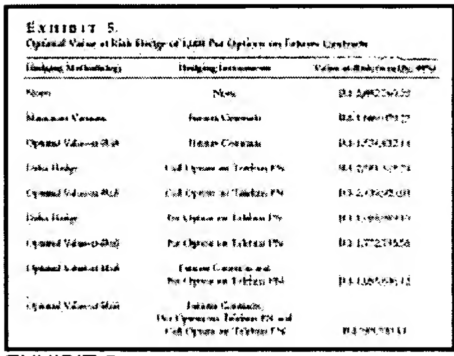

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EXHIBIT 5

EXHIBIT 6
Portfolio in the Brazilian Stock Market

Stocks	Portfolio Composition
Telebras PN	40%
Petrobras PN	20%
Vale PN	10%
Eletrobras PNB	10%
Cemig PN	10%
Usiminas PN	10%

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EXHIBIT 6

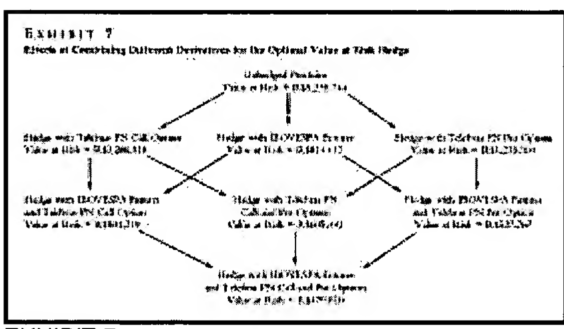

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EXHIBIT 7

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